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Dynamical gauge theory for the XY gauge glass model

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Abstract
Dynamical systems of gauge glass are investigated by the method of gauge transformation. Both stochastic and deterministic dynamics are treated. Several exact relations are derived from dynamical quantities such as equilibrium and nonequilibrium auto-correlation functions, relaxation functions of order parameter and internal energy. They provide physical properties in terms of dynamics in the SG phase, a possible mixed phase and the Griffiths phase, the multicritical dynamics and the aging phenomenon. We also have a plausible argument for the absence of re-entrant transition in two or higher dimensions.

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1. Introduction

The physics of disordered systems has been one of most fascinating subjects for theorists and experimentalists. The gauge glass is a typical example in this category and has attracted much attention. The model describes thermodynamics of various systems such as disordered magnets with random Dzyaloshinskii–Moriya interaction [1], Josephson junction arrays with positional disorder in a magnetic field [2], vortex glasses [3] and crystal systems on disordered substrates [4]. In three dimensions, the spin glass (SG) transition[1] for a strongly disordered regime has been confirmed in the gauge glass systems by numerical simulations [5–13] and renormalization group (RG) analyses [14, 15], which is consistent with experimental observations [16–18].

In two dimensions, there is a controversy about the existence of a glass-like phase in a strongly disordered regime. Although the long-range SG order is denied in two dimensions [19], there is a possibility of a quasi-long-range order in which the SG correlation decays according to a power law [20, 21] like the ferromagnetic (FM) correlation in the

[1] This disordered phase is sometimes called 'the glass phase' or 'the vortex glass phase'. However, in this paper, we call it the SG phase, since they have the same properties.

Kosterlitz–Thouless (KT) phase [22, 23]. Some numerical simulations supported the latter case [24–27], while experimental observation [28] and numerical simulations [9, 10, 13, 29, 30] deny it. In a weakly disordered regime, the instability of the KT phase against small disorder is suggested by the perturbation expansion and the RG analysis [31, 32], while it is denied by numerical simulations [4, 9, 25, 33] and other RG analyses [25, 34–37]. There is another controversy in two dimensions about the existence of a re-entrant transition from the KT phase to the non-KT one. Earlier works with real space RG analyses suggest re-entrance [1, 2, 38, 39], however, the analysis has been modified and provides for its absence [34–37]. The same results are obtained by Monte Carlo simulations [4, 9, 11, 25, 40, 41] and the RG analyses [42, 34].

Since randomness and frustration make it difficult to examine short-range systems analytically as well as numerically, only a few things have been confirmed definitely in spite of a large number of studies. Thus, it is highly desirable to have an analytic method for understanding the thermodynamic and critical behaviour in random spin systems. The method of gauge transformation is a useful technique to derive analytic results in gauge symmetric random spin systems [20, 21, 43–48]. It was firstly applied to the Ising SG models [43], and provides the internal energy and an upper bound on the specific heat exactly as non-singular functions of the temperature on the Nishimori line in the temperature–randomness phase diagram. Further, it is proved that the boundary between the FM and non-FM phases in the temperature lower than the multicritical point (MCP) at which paramagnetic (PM), FM and SG phases meet and the Nishimori line intersects, is vertical or re-entrant. Using a model with a modified probability distribution of randomness, the verticality is concluded [44], which implies the absence of re-entrance: although the theory needs a plausible but unproved assumption on the thermodynamic behaviour of the modified model, the result is consistent with other theories and experiments on Ising-like spin glasses [49]. The gauge transformation has been applied to gauge glasses with various symmetries [20] including the $U(1)$ symmetry, i.e. the XY gauge glass model. The same results as in the Ising case are derived. The absence of re-entrance is concluded for any dimensional model with a plausible assumption for the modified model, if the FM (or KT) phase is stable in the disordered regime. The gauge transformation is also extended to dynamical systems [45–48]. For the Ising SG case, some important relations are derived for nonequilibrium auto-correlation functions, order parameter and energy relaxations. They provide discussions on the absence of re-entrance, the regime of the mixed phase in the phase diagram, the multi-critical dynamics, and the equivalence of two nonequilibrium processes in the SG phase.

The dynamics is one of most important aspects in disordered systems because of the slow relaxation [49, 50]. In the case of standard SG theory, dynamical properties have been investigated for the mean field model [51], the Griffiths phase in short-range systems [52–55], the aging phenomenon [56–60] and so on. The aging phenomenon [61] is a typical realization of slow dynamics, especially for SG materials [62–65]. In contrast to the standard SG systems, not many studies on the dynamical properties have been performed for gauge glass systems. It would be helpful for future investigations to present exact results including dynamical properties.

In this paper, we study the gauge transformation for dynamical systems of the gauge glass model. In the static case, the gauge symmetric models can be treated in the same manner as the Ising SG [20]. It is also straightforward to apply the dynamical gauge theory for the Ising SG [45, 47, 48], in which the stochastic dynamics is treated, to the gauge glass model. Since the gauge glass system has a continuous spin space, one can consider a deterministic dynamics with equations of motion, which is also treated in the present theory. Several exact relations are derived among dynamical quantities such as equilibrium and nonequilibrium auto-correlation functions, relaxation functions of order parameter and internal energy. Using

them, we discuss physical properties including aging, multicritical dynamics, dynamics in the SG, a possible mixed phase and Griffiths phase and the absence of re-entrant transition in two or higher dimensions.

The organization of the paper is as follows. In section 2, the gauge glass model associated with the plane rotator is introduced. The probability distribution is chosen so that the system has the gauge symmetry. The results of the static gauge theory are summarized. In section 3, a stochastic dynamics is introduced using the master equation formalism, and some dynamical quantities are defined. The gauge transformation is introduced for this system. Some exact relations are derived for dynamical quantities. In section 4, physical properties are discussed by using the exact relations obtained. In section 5, equations of motion are introduced for the same gauge symmetric system, and the gauge transformation is applied. It is shown that the same relations can be derived for several choices of dynamics. The last section is devoted to the summary.

2. Gauge symmetric gauge glass model

The Hamiltonian we consider is

$$\mathcal{H} = J\tilde{\mathcal{H}}(\phi, \boldsymbol{\omega}) = -J \sum_{\langle ij \rangle} \cos(\phi_i - \phi_j + \omega_{ij}) \tag{2.1}$$

where $\phi_i, \omega_{ij} \in [-\pi, \pi)$. The set $\phi = (\phi_1, \phi_2, \ldots, \phi_N)$ represents a configuration of total N spins, and $\boldsymbol{\omega}$ represents a configuration of total N_{B} quenched random variables associated with each bond. The summation of $\langle ij \rangle$ is taken over all bonds; we make no restrictions on the type or the dimensions of the lattice, whereas one may suppose a usual nearest-neighbour interaction on the d-dimensional hypercubic lattice. The variable ω_{ij} has an odd property; $\omega_{ji} = -\omega_{ij}$.[2] For a particular bond configuration, the thermal equilibrium distribution at temperature $T = J/k_{\mathrm{B}}K$ is given by

$$\rho_{\mathrm{eq}}(\phi; K, \boldsymbol{\omega}) = \frac{\exp(-K\tilde{\mathcal{H}}(\phi; \boldsymbol{\omega}))}{Z(K; \boldsymbol{\omega})}. \tag{2.2}$$

The partition function is defined by

$$Z(K; \boldsymbol{\omega}) = \int \mathrm{d}\phi \exp(-K\tilde{\mathcal{H}}(\phi; \boldsymbol{\omega})) \tag{2.3}$$

where the integration of $\mathrm{d}\phi$ denotes the multiple integrations for the N spin variables ϕ_i. The probability distribution for randomness is taken as

$$P(\boldsymbol{\omega}; D) = \prod_{\langle ij \rangle} \frac{\exp(D \cos \omega_{ij})}{2\pi I_0(D)} = \frac{\exp(-D\tilde{\mathcal{H}}(\mathbf{F}; \boldsymbol{\omega}))}{Y(D)} \tag{2.4}$$

where $\mathbf{F} = (0, 0, \ldots, 0)$ represents an all-aligned spin state, $I_n(D)$ is the modified Bessel function, and

$$Y(D) \equiv (2\pi I_0(D))^{N_{\mathrm{B}}}. \tag{2.5}$$

In this distribution, which is central to the present paper (see also figure 2), the variable D controls the strength of randomness; $D = 0$ and ∞ correspond to the full random case and the non-random case, respectively. The random average is denoted by

$$[\cdots]_{\mathrm{c}} = \int \mathrm{d}\boldsymbol{\omega} P(\boldsymbol{\omega}; D) \cdots. \tag{2.6}$$

[2] The sign of ω_{ij} in equation (2.1) is not important, since the distribution of ω_{ij} (2.4) is an even function.

The integration of $\mathrm{d}\boldsymbol{\omega}$ denotes the multiple integrations for the N_B random variables ω_{ij}. In the following, for simplicity, we will sometimes omit the dependence on spin set $\boldsymbol{\phi}$, bond set $\boldsymbol{\omega}$, inverse temperature K and/or randomness D from functions defined above, if they are trivial or unimportant.

The gauge transformations for functions of $\boldsymbol{\phi}$ and $\boldsymbol{\omega}$ are defined by

$$U_{\boldsymbol{\psi}} : \phi_i \longrightarrow \phi_i - \psi_i \tag{2.7}$$

$$V_{\boldsymbol{\psi}} : \omega_{ij} \longrightarrow \omega_{ij} + \psi_i - \psi_j \tag{2.8}$$

where $\boldsymbol{\psi} = (\psi_1, \psi_2, \ldots, \psi_N)$ is an arbitrary state of N spins. Each set of transformations forms a group. While variables ϕ_i and ω_{ij} possibly take values out of the domain, $[-\pi, \pi)$, in this transformation, we always identify them with a residue of 2π. The Hamiltonian (2.1) is invariant under the transformation $U_{\boldsymbol{\psi}} V_{\boldsymbol{\psi}}$:

$$U_{\boldsymbol{\psi}} V_{\boldsymbol{\psi}} \tilde{\mathcal{H}}(\boldsymbol{\phi}; \boldsymbol{\omega}) = \tilde{\mathcal{H}}(\boldsymbol{\phi}; \boldsymbol{\omega}). \tag{2.9}$$

The gauge transformation of the distribution (2.4) is given by

$$V_{\boldsymbol{\psi}} P(\boldsymbol{\omega}; D) = \frac{\exp(-D\tilde{\mathcal{H}}(\boldsymbol{\psi}; \boldsymbol{\omega}))}{Y(D)} = \frac{Z(D; \boldsymbol{\omega})}{Y(D)} \rho_{\mathrm{eq}}(\boldsymbol{\psi}; D, \boldsymbol{\omega}) \tag{2.10}$$

where we note that the spin set in $\tilde{\mathcal{H}}$ and the temperature are different from the usual ones. Another important property in the theory is the invariant integral for $\boldsymbol{\phi}$ and $\boldsymbol{\omega}$:

$$\int \mathrm{d}\boldsymbol{\phi} \cdots = \int \mathrm{d}\boldsymbol{\phi}\, U_{\boldsymbol{\psi}} \cdots \tag{2.11}$$

$$\int \mathrm{d}\boldsymbol{\omega} \cdots = \int \mathrm{d}\boldsymbol{\omega}\, V_{\boldsymbol{\psi}} \cdots . \tag{2.12}$$

Using equations (2.9) and (2.10), one can show [20] that the partition function is gauge invariant under $V_{\boldsymbol{\psi}}$,

$$V_{\boldsymbol{\psi}} Z(K; \boldsymbol{\omega}) = Z(K; \boldsymbol{\omega}) \tag{2.13}$$

providing the invariance of thermal distribution

$$U_{\boldsymbol{\psi}} V_{\boldsymbol{\psi}}\, \rho_{\mathrm{eq}}(\boldsymbol{\phi}; K, \boldsymbol{\omega}) = \rho_{\mathrm{eq}}(\boldsymbol{\phi}; K, \boldsymbol{\omega}). \tag{2.14}$$

Then, we obtain the gauge invariance of the averaged energy

$$V_{\boldsymbol{\psi}} \langle \tilde{\mathcal{H}} \rangle_K = \langle \tilde{\mathcal{H}} \rangle_K . \tag{2.15}$$

Note that we will use the terminology 'gauge invariant' only for functions of $\boldsymbol{\omega}$ invariant under $V_{\boldsymbol{\psi}}$. Since the lhs of equation (2.12) is independent of $\boldsymbol{\psi}$, it is convenient to derive another expression of random average instead of equation (2.6):

$$[\cdots]_{\mathrm{c}} = \int \mathrm{d}\boldsymbol{\omega}\, \mathrm{d}\boldsymbol{\psi} \frac{Z(D; \boldsymbol{\omega})}{(2\pi)^N Y(D)} \rho_{\mathrm{eq}}(\boldsymbol{\psi}; D, \boldsymbol{\omega}) V_{\boldsymbol{\psi}} \cdots . \tag{2.16}$$

The random average of gauge invariant function $Q(\boldsymbol{\omega})$ can be expressed as

$$[Q(\boldsymbol{\omega})]_{\mathrm{c}} = \int \mathrm{d}\boldsymbol{\omega} \frac{Z(D; \boldsymbol{\omega})}{(2\pi)^N Y(D)} Q(\boldsymbol{\omega}). \tag{2.17}$$

In three or higher dimensions, the topology of the K–D phase diagram is expected as in figure 1, where PM, FM and SG phases appear. A possible Griffiths phase is also indicated. On the line $K = D$, this phase and the FM phase would be separated by the multicritical point (MCP) already mentioned. In two dimensions, the KT phase appears instead of the FM phase. The SG phase disappears or becomes a possible KT-like SG phase. The dashed line $K = D$ is called the Nishimori line around which the following properties have been found from the

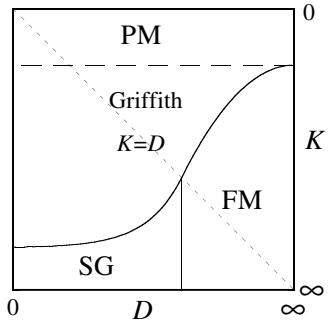

Figure 1. Typical phase diagram of gauge glass model in the D–K plane (D controls the randomness and $K = J/k_{\mathrm{B}}T$). The dashed line $K = D$ is the Nishimori line. The FM phase boundary in the low temperature region is assumed to be vertical. A possible regime of the Griffiths phase is indicated.

static gauge theory [20, 46]:

- The energy and the upper bound of the specific heat are expressed by analytic functions of temperature;

$$E(D, D) = -J\frac{\mathrm{d}}{\mathrm{d}D}\ln Y(D) = -N_{\mathrm{B}}J\frac{I_1(D)}{I_0(D)} \tag{2.18}$$

$$C(D, D) \leqslant k_{\mathrm{B}}D^2\frac{\mathrm{d}^2}{\mathrm{d}D^2}\ln Y(D). \tag{2.19}$$

- The FM and SG order parameters coincide with each other;

$$m(D, D) = q(D, D). \tag{2.20}$$

  Thus, no SG phase appears on this line, since $m = 0$ in the SG phase.
- The line is likely to intersect the MCP of PM, FM and SG phases.
- The phase boundary between the FM (or KT) and the SG (or PM) phases below this line must be vertical to the D-axis or re-entrant to the large D side [20]. A plausible argument can be made for the verticality.

It is remarked that the probability distribution function (2.4) is different from the usual Gaussian form, which has been studied in many cases. This is because we need a gauge symmetric form in the theory, which restricts the distribution function relating to the Hamiltonian as in equation (2.4). It is shown that the same properties can also be proved for another gauge symmetric $U(1)$ model, the Villain model [66] with Gaussian random gauge, in which the local Boltzmann factor is given by

$$\mathrm{e}^{V(\phi_i-\phi_j+\omega_{ij})} = \sum_{n=-\infty}^{\infty}\mathrm{e}^{-K(\phi_i-\phi_j+\omega_{ij}-2\pi n)^2/2}. \tag{2.21}$$

We consider that physical properties are equivalent in these gauge glass models irrespective of the detail of distribution P. In fact, as seen in figure 2, the shape of the function (2.4) is quite similar to the Gaussian. As will be seen later, the present theory can be applied to the Villain model in the case of the stochastic dynamics in section 3. However, we proceed with our present theory by the rotator model. It is somehow difficult to apply in the case of the deterministic dynamics, since the explicit form of the Hamiltonian, which is necessary in the equations of motion, is not unique.

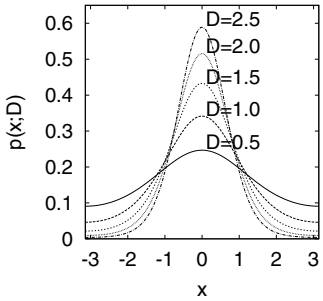

Figure 2. The probability distribution function for one bond $p(x;\, D) \equiv \exp(D\cos x)/(2\pi I_0(D))$.

3. Stochastic dynamics

First, we consider the stochastic dynamics following the results of Ising SG [45, 47, 48].

3.1. Master equation

We consider a Markov process for a fixed bond configuration, in which the density of state evolves with the master equation [67]

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_t(\phi) = \int \mathrm{d}\phi'\{W(\phi|\phi')\rho_t(\phi') - W(\phi'|\phi)\rho_t(\phi)\}. \tag{3.1}$$

The transition probability $W(\phi|\phi')$ is non-negative and satisfies the condition of the detailed balance

$$W(\phi|\phi')\rho_{\mathrm{eq}}(\phi';\, K, \boldsymbol{\omega}) = W(\phi'|\phi)\rho_{\mathrm{eq}}(\phi;\, K, \boldsymbol{\omega}) \tag{3.2}$$

which guarantees the stability of the equilibrium distribution ρ_{eq}. Condition (3.2) is automatically satisfied if one uses the expression with a symmetric matrix $w(\phi|\phi')$:

$$W(\phi|\phi') = \frac{w(\phi|\phi')}{\rho_{\mathrm{eq}}(\phi')}. \tag{3.3}$$

The function $w(\phi|\phi')$ depends on the details of dynamics considered. As in [45], I assume

$$w(\phi|\phi') = w_0\delta_1[\phi, \phi']\rho_{\mathrm{eq}}(\phi)^{\Theta(\Delta[\phi,\phi'])}\rho_{\mathrm{eq}}(\phi')^{\Theta(\Delta[\phi',\phi])} \tag{3.4}$$

(where $\Theta(x)$ denotes the Heaviside function) for the single-spin-flip Metropolis dynamics [68], and

$$w(\phi|\phi') = w_0\sum_i \frac{\delta_{1i}[\phi, \phi']\rho_{\mathrm{eq}}(\phi)\rho_{\mathrm{eq}}(\phi')}{\int \mathrm{d}\phi''\delta_{1i}[\phi'', \phi']\rho_{\mathrm{eq}}(\phi'')} \tag{3.5}$$

for the single-spin-flip heat-bath dynamics, where

$$\Delta[\phi, \phi'] \equiv \tilde{\mathcal{H}}(\phi;\, \boldsymbol{\omega}) - \tilde{\mathcal{H}}(\phi';\, \boldsymbol{\omega}) \tag{3.6}$$

is the energy difference of two states

$$\delta_1[\phi, \phi'] \equiv \sum_i \delta_{1i}[\phi, \phi'] \tag{3.7}$$

$$\delta_{1i}[\phi, \phi'] \equiv \prod_{j\neq i} \delta(\phi_j - \phi'_j) \tag{3.8}$$

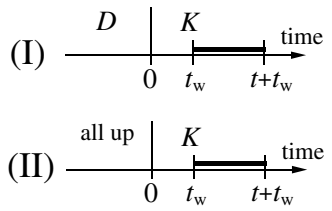

Figure 3. Illustrations of two nonequilibrium processes, I and II. Between t_w and $t + t_w$, the correlation is measured.

which are the single spin-flip and the ith spin-flip operators, respectively. Note that the denominator of the summand in equation (3.5) is unchanged if the variable ϕ' in the argument of $\delta_{1i}[\phi'', \phi']$ is replaced by ϕ providing the symmetry of $w(\phi|\phi')$.

Since the diagonal element plays no role in either equation (3.1) or (3.2), one may define it in such a way as

$$W(\phi|\phi') = \frac{w(\phi|\phi')}{\rho_{\mathrm{eq}}(\phi')} - \delta(\phi - \phi') \int \mathrm{d}\phi'' \frac{w(\phi''|\phi)}{\rho_{\mathrm{eq}}(\phi)}. \tag{3.9}$$

Then, the master equation (3.1) is reduced to

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_t(\phi) = \int \mathrm{d}\phi' W(\phi|\phi')\rho_t(\phi') \tag{3.10}$$

and the solution is given by

$$\rho_t(\phi) = \int \mathrm{d}\phi' \langle\phi|\, \mathrm{e}^{t\mathbf{W}}|\phi'\rangle \rho_0(\phi'). \tag{3.11}$$

The function $\langle\phi|\, \mathrm{e}^{t\mathbf{W}}|\phi'\rangle$ is the integration kernel for time evolution defined by

$$\langle\phi|\, \mathrm{e}^{t\mathbf{W}}|\phi'\rangle \equiv \sum_{n=0}^{\infty} \frac{t^n}{n!} \langle\phi|\mathbf{W}^n|\phi'\rangle \tag{3.12}$$

$$\langle\phi|\mathbf{W}^n|\phi'\rangle \equiv \int \left(\prod_{\ell=1}^{n-1} \mathrm{d}\phi^{(\ell)}\right) \prod_{k=0}^{n-1} W(\phi^{(k+1)}|\phi^{(k)}) \tag{3.13}$$

where $\phi^{(n)} = \phi$ and $\phi^{(0)} = \phi'$. It has been proved that all eigenvalues of the operator $\mathbf{W}$ with (3.2) are real and negative except the zero eigenvalue corresponding to the equilibrium distribution [67]. Thus, all solutions with any initial conditions tend to the equilibrium distribution as $t \to \infty$.

3.2. Dynamical processes

We examine two nonequilibrium processes I and II in figure 3. Process I is related to the zero-field cooling (ZFC). At the initial time $t = 0$, the system is kept in the equilibrium state with a temperature D and zero field; hereafter, we use the terminology 'temperature' for K and D instead of 'inverse temperature'. The temperature is immediately changed (usually quenched) and the system relaxes in a heat bath with another temperature K with $t > 0$. The average for dynamical ensembles in this process is denoted by $\langle\cdots\rangle_K^D$. Process II is related to the field quench (FQ) [59, 63]. The system starts from an all-aligned state $\mathbf{F}$ at $t = 0$ and relaxes in the same heat bath as in process I with $t > 0$; the average is denoted by $\langle\cdots\rangle_K^{\mathbf{F}}$. Since the all-aligned state provides the strong-field limit, this represents the process with a

field quenched from ∞ to zero at $t = 0$. Note that the FQ is not equivalent to the field cooling in which the applied field is weaker and is quenched after waiting.

We define the waiting-time dependence of nonequilibrium auto-correlation functions [59, 60] for processes I and II. In the language of the master equation, that for process I is expressed as

$$\langle \mathrm{e}^{\mathrm{i}\phi_i(t+t_\mathrm{w})-\mathrm{i}\phi_i(t_\mathrm{w})}\rangle_K^D = \int \mathrm{d}\phi\,\mathrm{d}\phi'\,\mathrm{d}\phi_0 \langle\phi|\,\mathrm{e}^{t\mathbf{W}}|\phi'\rangle\langle\phi'|\,\mathrm{e}^{t_\mathrm{w}\mathbf{W}}|\phi_0\rangle\rho_\mathrm{eq}(\phi_0;\,D)\,\mathrm{e}^{\mathrm{i}\phi_i-\mathrm{i}\phi_i'}. \tag{3.14}$$

When $K = D$, the average $\langle\cdots\rangle_D^D$ represents the dynamical average in the equilibrium process $\langle\cdots\rangle_D^\mathrm{eq}$. In this case, the auto-correlation function (3.14) becomes independent of t_w since the thermal distribution $\rho_\mathrm{eq}(\phi;\,K,\omega)$ is an eigenfunction of $W(\phi'|\phi)$, i.e.

$$\begin{aligned}\langle \mathrm{e}^{\mathrm{i}\phi_i(t+t_\mathrm{w})-\mathrm{i}\phi_i(t_\mathrm{w})}\rangle_D^D &= \langle \mathrm{e}^{\mathrm{i}\phi_i(t)-\mathrm{i}\phi_i(0)}\rangle_D^\mathrm{eq} \\ &= \int \mathrm{d}\phi\,\mathrm{d}\phi'\langle\phi|\,\mathrm{e}^{t\mathbf{W}}|\phi'\rangle\rho_\mathrm{eq}\,(\phi';\,K)\,\mathrm{e}^{\mathrm{i}\phi_i-\mathrm{i}\phi_i'}.\end{aligned} \tag{3.15}$$

If one defines an auto-correlation function for the process relaxed from a fixed state ψ—let us call this process II′—

$$\langle \mathrm{e}^{\mathrm{i}\phi_i(t+t_\mathrm{w})-\mathrm{i}\phi_i(t_\mathrm{w})}\rangle_K^\psi = \int \mathrm{d}\phi\,\mathrm{d}\phi'\langle\phi|\,\mathrm{e}^{t\mathbf{W}}|\phi'\rangle\langle\phi'|\,\mathrm{e}^{t_\mathrm{w}\mathbf{W}}|\psi\rangle\,\mathrm{e}^{\mathrm{i}\phi_i-\mathrm{i}\phi_i'} \tag{3.16}$$

that for process II is expressed as $\langle \mathrm{e}^{\mathrm{i}\phi_i(t+t_\mathrm{w})-\mathrm{i}\phi_i(t_\mathrm{w})}\rangle_K^\mathbf{F}$, and equation (3.14) is rewritten as

$$\langle \mathrm{e}^{\mathrm{i}\phi_i(t+t_\mathrm{w})-\mathrm{i}\phi_i(t_\mathrm{w})}\rangle_K^D = \int \mathrm{d}\psi\,\rho_\mathrm{eq}(\psi;\,D)\langle \mathrm{e}^{\mathrm{i}\phi_i(t+t_\mathrm{w})-\mathrm{i}\phi_i(t_\mathrm{w})}\rangle_K^\psi. \tag{3.17}$$

In the long-time limit ($t \to \infty$), the thermodynamic limit of $\left[\langle \mathrm{e}^{\mathrm{i}\phi_i(t)-\mathrm{i}\phi_i(0)}\rangle_K^\mathrm{eq}\right]_\mathrm{c}$ converges to the SG order parameter q_EA, which is non-vanishing in FM or SG phases [49, 50]. If one sets $t_\mathrm{w} = 0$, equation (3.16) with $\psi = \mathbf{F}$ becomes

$$\langle \mathrm{e}^{\mathrm{i}\phi_i(t)}\rangle_K^\mathbf{F} = \int \mathrm{d}\phi\langle\phi|\,\mathrm{e}^{t\mathbf{W}}|\mathbf{F}\rangle\,\mathrm{e}^{\mathrm{i}\phi_i} \tag{3.18}$$

which is the FM order parameter at time t relaxed from the complete FM state, $\mathbf{F}$, at $t = 0$. When $t \to \infty$, the thermodynamic limit of $\left[\langle \mathrm{e}^{\mathrm{i}\phi_i(t)}\rangle_K^\mathbf{F}\right]_\mathrm{c}$ approaches the spontaneous ordering. Note that all quantities defined above containing the exponential $\mathrm{e}^{\mathrm{i}\phi_i}$ have real values in the average because of the spin–flip symmetry, $\phi_i \to \phi_i + \pi$, for the Hamiltonian and the matrix $W(\phi|\phi')$.

Other relaxation functions, the nonequilibrium relaxations of exchange energy, are defined as

$$\langle\mathcal{H}(t)\rangle_K^D = J\int \mathrm{d}\phi\,\mathrm{d}\phi'\langle\phi|\,\mathrm{e}^{t\mathbf{W}}|\phi'\rangle\rho_\mathrm{eq}(\phi';\,D,\omega)\tilde{\mathcal{H}}(\phi;\,\omega) \tag{3.19}$$

$$\langle\mathcal{H}(t)\rangle_K^\mathbf{F} = J\int \mathrm{d}\phi\langle\phi|\,\mathrm{e}^{t\mathbf{W}}|\mathbf{F}\rangle\tilde{\mathcal{H}}(\phi;\,\omega). \tag{3.20}$$

3.3. Gauge transformation

Let us examine the gauge transformation of the functions defined above. We introduce the transformation for ϕ':

$$U'_\psi : \phi_i' \longrightarrow \phi_i' - \psi_i \tag{3.21}$$

to show the invariance

$$U_\psi U'_\psi V_\psi\langle\phi|\,\mathrm{e}^{t\mathbf{W}}|\phi'\rangle = \langle\phi|\,\mathrm{e}^{t\mathbf{W}}|\phi'\rangle. \tag{3.22}$$

From equations (2.9)–(2.13), it is easy to see that $\rho_{\rm eq}(\phi; K, \boldsymbol{\omega})$, $\delta_{1i}[\phi, \phi']$ and $\Delta[\phi, \phi']$ are invariant under $U_\psi U'_\psi V_\psi$. This yields the invariance

$$U_\psi U'_\psi V_\psi w(\phi|\phi') = w(\phi|\phi') \tag{3.23}$$

in the case of the Metropolis (3.4) and the heat-bath dynamics (3.5). It is natural to restrict the dynamics in such a way as equation (3.23) holds. This yields the invariance

$$U_\psi U'_\psi V_\psi W(\phi|\phi') = W(\phi|\phi') \tag{3.24}$$

where the invariant integral for ϕ'' like equation (2.11) is necessary to derive the case $\phi = \phi'$. Let us consider term by term in the expansion of $\langle\phi|\,{\rm e}^{t\mathbf{W}}|\phi'\rangle$, equation (3.13). Similarly to ϕ and ϕ', the gauge transformations for $\phi^{(k)}(k = 1, 2, \ldots, n-1)$ are introduced as

$$U_\psi^{(k)} : \phi_i^{(k)} \longrightarrow \phi_i^{(k)} - \psi_i. \tag{3.25}$$

Then, equation (3.24) with the invariant integral for $\phi^{(k)}$ like (2.11) leads to

$$\begin{aligned} U_\psi U'_\psi V_\psi \langle\phi|\mathbf{W}^n|\phi'\rangle &= \int \left(\prod_{\ell=1}^{n-1} {\rm d}\phi^{(\ell)}\, U_\psi^{(\ell)}\right) U_\psi U'_\psi V_\psi \prod_{k=0}^{n-1} W(\phi^{(k+1)}|\phi^{(k)}) \\ &= \int \left(\prod_{\ell=1}^{n-1} {\rm d}\phi^{(\ell)}\right) \prod_{k=0}^{n-1} W(\phi^{(k+1)}|\phi^{(k)}) \end{aligned} \tag{3.26}$$

providing the invariance (3.22).

From equations (2.14) and (3.22) with the invariant integral (2.11) taken into account, one can show that the auto-correlation function for process I is gauge invariant as

$$V_\psi \langle {\rm e}^{{\rm i}\phi_i(t) - {\rm i}\phi_i(0)}\rangle_K^{\rm eq} = \langle {\rm e}^{{\rm i}\phi_i(t) - {\rm i}\phi_i(0)}\rangle_K^{\rm eq} \tag{3.27}$$

$$V_\psi \langle {\rm e}^{{\rm i}\phi_i(t+t_{\rm w}) - {\rm i}\phi_i(t_{\rm w})}\rangle_K^{D} = \langle {\rm e}^{{\rm i}\phi_i(t+t_{\rm w}) - {\rm i}\phi_i(t_{\rm w})}\rangle_K^{D} \tag{3.28}$$

while that for process II is transformed as

$$\begin{aligned} V_\psi \langle {\rm e}^{{\rm i}\phi_i(t+t_{\rm w}) - {\rm i}\phi_i(t_{\rm w})}\rangle_K^{\mathbf{F}} &= \int {\rm d}\phi\, {\rm d}\phi' U_\psi U'_\psi V_\psi \langle\phi|\,{\rm e}^{t\mathbf{W}}|\phi'\rangle\langle\phi'|\,{\rm e}^{t_{\rm w}\mathbf{W}}|\mathbf{F}\rangle\, {\rm e}^{{\rm i}\phi_i - {\rm i}\phi_i'} \\ &= \langle {\rm e}^{{\rm i}\phi_i(t+t_{\rm w}) - {\rm i}\phi_i(t_{\rm w})}\rangle_K^{\psi} \end{aligned} \tag{3.29}$$

where we use the invariant integral (3.11), and

$$U_\psi V_\psi \langle\phi|\,{\rm e}^{t\mathbf{W}}|\phi'\rangle = U'_{-\psi}\langle\phi|\,{\rm e}^{t\mathbf{W}}|\phi'\rangle = \langle\phi|\,{\rm e}^{t\mathbf{W}}|\phi' + \psi\rangle \tag{3.30}$$

obtained from equation (3.22). The variables $-\psi$ and $\phi' + \psi$ represent the spin states, $(-\psi_1, -\psi_2, \ldots, -\psi_N)$ and $(\phi_1' + \psi_1, \phi_2' + \psi_2, \ldots, \phi_N' + \psi_N)$, respectively. Using equations (2.16), (3.17) and (3.29), we derive

$$\left[\langle {\rm e}^{{\rm i}\phi_i(t+t_{\rm w}) - {\rm i}\phi_i(t_{\rm w})}\rangle_K^{\mathbf{F}}\right]_{\rm c} = \int {\rm d}\boldsymbol{\omega}\, \frac{Z(D;\boldsymbol{\omega})}{(2\pi)^N Y(D)} \langle {\rm e}^{{\rm i}\phi_i(t+t_{\rm w}) - {\rm i}\phi_i(t_{\rm w})}\rangle_K^{D}. \tag{3.31}$$

Because of the gauge invariance (3.28), equation (3.31) is rewritten by equation (2.17) as

$$\left[\langle {\rm e}^{{\rm i}\phi_i(t+t_{\rm w}) - {\rm i}\phi_i(t_{\rm w})}\rangle_K^{\mathbf{F}}\right]_{\rm c} = \left[\langle {\rm e}^{{\rm i}\phi_i(t+t_{\rm w}) - {\rm i}\phi_i(t_{\rm w})}\rangle_K^{D}\right]_{\rm c} \tag{3.32}$$

for any waiting time $t_{\rm w}$, time interval t, temperature K and degree of randomness D. We call it the 'aging relation', since it relates the aging phenomena in two distinct processes whatever waiting-time dependence is essential for the aging. From the invariance (2.9), a similar relation can be derived for the energy defined in equations (3.19) and (3.20):

$$\left[\langle\mathcal{H}(t)\rangle_K^{\mathbf{F}}\right]_{\rm c} = \left[\langle\mathcal{H}(t)\rangle_K^{D}\right]_{\rm c}. \tag{3.33}$$

4. Physical properties

The aging relation (3.32) and equation (3.33) include parameters K and D characterizing the relaxation in $t > 0$. For each phase, one can examine the physical meaning of the relation by choosing (K, D) appropriately. Note that the initial temperature of process I is always located on $K = D$.

4.1. Dynamics in the SG phase

First, let us consider the region with sufficiently strong randomness (small D) and sufficiently low temperatures (large K), where the SG phase appears if it exists. In such a region, D indicates a high temperature (small D), and the relaxations $\left[\langle \mathrm{e}^{\mathrm{i}\phi_i(t+t_\mathrm{w})-\mathrm{i}\phi_i(t_\mathrm{w})}\rangle_K^D\right]_\mathrm{c}$ and $\left[\langle \mathcal{H}(t)\rangle_K^D\right]_\mathrm{c}$ for process I describe a ZFC process (cooled immediately from a high temperature with D). On the other hand, $\left[\langle \mathrm{e}^{\mathrm{i}\phi_i(t+t_\mathrm{w})-\mathrm{i}\phi_i(t_\mathrm{w})}\rangle_K^\mathbf{F}\right]_\mathrm{c}$ and $\left[\langle \mathcal{H}(t)\rangle_K^\mathbf{F}\right]_\mathrm{c}$ for process II are regarded as the relaxations in an FQ process (demagnetization from the strong field limit). Thus, relations (3.32) and (3.33) imply the equivalence of the dynamics especially the aging phenomenon in the ZFC and FQ processes. The same relation was pointed out in real SG materials [63, 65]. At a glance, this relation expresses a trivial fact, since both initial states are located in the PM phase and both systems relax into the SG phase. However, the equivalence at any waiting time, which means the equivalence of the dynamical structure at any stage of relaxation, is non-trivial.

In the case of sufficiently weak randomness (large D), the initial states of both nonequilibrium processes represent almost the same spin state (complete FM state). Then, equations (3.32) and (3.33) show a proper phenomenon.

4.2. Dynamics on $K = D$

Next, we examine the dynamics on the line $K = D$, where the dynamical average $\langle\cdots\rangle_K^D$ becomes the average in the equilibrium process $\langle\cdots\rangle_D^\mathrm{eq}$. From equations (3.15) and (3.32), we derive

$$\left[\langle \mathrm{e}^{\mathrm{i}\phi_i(t+t_\mathrm{w})-\mathrm{i}\phi_i(t_\mathrm{w})}\rangle_D^\mathbf{F}\right]_\mathrm{c} = \left[\langle \mathrm{e}^{\mathrm{i}\phi_i(t)-\mathrm{i}\phi_i(0)}\rangle_D^\mathrm{eq}\right]_\mathrm{c}. \tag{4.1}$$

If one sets $t_\mathrm{w} = 0$, equation (4.1) yields

$$\left[\langle \mathrm{e}^{\mathrm{i}\phi_i(t)}\rangle_D^\mathbf{F}\right]_\mathrm{c} = \left[\langle \mathrm{e}^{\mathrm{i}\phi_i(t)-\mathrm{i}\phi_i(0)}\rangle_D^\mathrm{eq}\right]_\mathrm{c} \tag{4.2}$$

which implies the equivalence of FM and SG orderings (if one takes the limit of $t \to \infty$). Since, in the SG phase, the SG order remains finite whereas the FM order disappears, equation (4.2) indicates the absence of the SG phase on the line $K = D$ as in the analysis of the static orderings (2.20) [20].

Equations (4.1) and (4.2) also imply the equivalence of the equilibrium and nonequilibrium relaxations. The internal energy of the initial state $\mathbf{F}$ is identical to that on $K = D$, which is equal to the equilibrium energy on $K = D$ (2.18) [20]. It is easy to see that the rhs of equation (3.33) is independent of time t if $K = D$:

$$\left[\langle \mathcal{H}(t)\rangle_D^\mathbf{F}\right]_\mathrm{c} = -N_\mathrm{B} J \frac{I_1(D)}{I_0(D)}. \tag{4.3}$$

Since the equilibrium energy is an increasing function of the temperature, we expect that $\left[\langle \mathcal{H}(t)\rangle_K^\mathbf{F}\right]_\mathrm{c}$ is an increasing function of t above $K = D$ and a decreasing function below it.

As seen from equations (4.1) and (4.2), the auto-correlation function in the FQ process is independent of the waiting time, which suggests the absence of aging on $K = D$. In the

region below the MCP on $K = D$, where the spontaneous ordering appears, it is not clear that the FQ process is appropriate for the observation of the aging, since the initial all-aligned state is not so different from the final equilibrium state with broken symmetry. While the rotational symmetry is initially broken in the FQ process, the dynamical behaviour is equivalent to that in the equilibrium (symmetric) process—see equation (4.1). Thus, we assume that the FQ process exhibits an aging feature even in such a symmetry-broken region.

4.3. Mixed phase and Griffiths phase

Aging has been considered an inherent property in the SG phase from experimental [62–65] and theoretical [56–60] viewpoints. Since the aging phenomenon is a typical observation of the complex phase space for slow dynamics, it could occur in other complex phases. In such a phase, aging would also be inherent, which means that it occurs in the whole of the phase when it is observed in some parts of the phase.

Following the results of the SK model and experimental observations [49] for the SG phenomenon, one may consider a possibility of a mixed phase between the FM and SG phases even in the gauge glass systems. It is natural to consider that aging occurs in the mixed phase, since the SG feature in the mixed phase provides typical slow dynamics which reveals aging. If aging is inherent in the mixed phase, this indicates that the line $K = D$ does not enter the mixed phase. This restricts the topology of the phase diagram: it must appear below $K = D$ if it even exists.

In the Ising SG systems, it has also been pointed out that there exists a dynamically singular phase called the Griffiths phase [69] between the critical temperature of the pure system and the phase boundary of the low temperature phase (the FM or the SG) [52–55]—see figure 1. The possibility of the Griffiths phase has not yet been discussed much in the gauge glass systems. Figure 1 is proposed just following an analogy with the dilute ferromagnet [69]. Concerning the Griffiths phase, two cases can be considered to understand the present result if aging is an inherent property: (a) *There is no aging in the whole of the Griffiths phase*, if the line $K = D$ intersects the Griffiths phase as in figure 1. (b) *There is no Griffiths phase at least around $K = D$*. In the former case, even if slow dynamics is observed in the Griffiths phase, it is quite different from that in the SG phase [70]. The latter case allows the existence of the Griffiths phase below $K = D$. Another possibility is that (c) *the aging is not inherent at least in the Griffiths phase*. This means that aging occurs in some parts of the phase. Although it is not obvious which is correct in the present framework, the above result restricts the existence of the aging phenomenon and the region of the Griffiths phase, and supports their future investigation.

4.4. Multicritical dynamics

Let us consider the critical relaxation at the MCP. At the FM critical point, in general, the FM order parameter decays in a power law asymptotically as

$$m(t) \equiv \left[\langle \mathrm{e}^{\mathrm{i}\phi_i(t)}\rangle_K^{\mathbf{F}}\right]_{\mathrm{c}} \sim t^{-\lambda_m}. \tag{4.4}$$

Then, the dynamic scaling hypothesis [71]

$$m(t, \varepsilon, L) = L^{-\beta/\nu}\bar{m}(L^{1/\nu}\varepsilon, L^{-z}t) \qquad \varepsilon = (K_{\mathrm{c}} - K)/K_{\mathrm{c}} \tag{4.5}$$

reveals the relation of exponents

$$\lambda_m = \frac{\beta}{z\nu}. \tag{4.6}$$

On the other hand, the equilibrium auto-correlation function decays in a power law as

$$q(t) \equiv \left[\langle \mathrm{e}^{\mathrm{i}\phi_i(t)-\mathrm{i}\phi_i(0)}\rangle_K^{\mathrm{eq}}\right]_{\mathrm{c}} \sim t^{-\lambda_q} \tag{4.7}$$

at both FM and SG critical points. At the FM critical point, the exponent λ_q is associated with the FM criticality; a similar dynamic scaling reveals

$$\lambda_q = \frac{2\beta}{z\nu}. \tag{4.8}$$

Thus, λ_m and λ_q are twice different in the FM case. At the SG critical point, it is associated with the SG criticality; the dynamic scaling hypothesis for the SG ordering

$$q(t, \varepsilon, L) = L^{-\tilde{\beta}/\tilde{\nu}}\bar{q}(L^{1/\tilde{\nu}}\varepsilon, L^{-\tilde{z}}t) \tag{4.9}$$

where tilde exponents are those for SG ordering, yields

$$\lambda_q = \frac{\tilde{\beta}}{\tilde{z}\tilde{\nu}}. \tag{4.10}$$

The scaling field is not unique in the random case. One can define another scaling form instead of (4.5) with another scaling field such as $(D_{\mathrm{c}} - D)/D_{\mathrm{c}}$. At the MCP, the scaling field ε can be chosen both as $\varepsilon = (D_{\mathrm{c}} - D)/D_{\mathrm{c}}$ and as $\varepsilon = (K_{\mathrm{c}} - K)/K_{\mathrm{c}}$ since they are of the same order around the MCP on $K = D$. Note that critical exponents β and ν depend on the way of approaching the critical point. Thus, they are defined along $K = D$ in equations (4.5) and (4.6) for the MCP case. On the other hand, the ratio β/ν and the exponent z are independent of the scaling field but depend on the critical point itself. On $K = D$, we have derived the relation (4.2). This gives

$$L^{-\beta/\nu}\bar{m}(L^{1/\nu}\varepsilon, L^{-z}t) = L^{-\tilde{\beta}/\tilde{\nu}}\bar{q}(L^{1/\tilde{\nu}}\varepsilon, L^{-\tilde{z}}t) \tag{4.11}$$

for any L, t and ε (K or D) providing

$$\lambda_m = \lambda_q \tag{4.12}$$

$$z = \tilde{z} \tag{4.13}$$

$$\beta/\nu = \tilde{\beta}/\tilde{\nu}. \tag{4.14}$$

Note that the line $K = D$ does not enter the SG phase but just touches it at the MCP [20]. The dynamical exponent $\tilde{z}$ and the ratio $\tilde{\beta}/\tilde{\nu}$ are independent of the way of approaching the criticality, and characterized by the criticality itself. Therefore, exponents $\tilde{z}$ and $\tilde{\beta}/\tilde{\nu}$ are associated with the SG ordering at the MCP. Equations (4.12)–(4.14) relate exponents of FM ordering and SG ordering at the MCP. The relations for FM dynamic exponents, (4.6) and (4.8), would keep along the phase boundary from the pure case up to the MCP in figure 1, and so does the relation for SG ones (4.10) from $D = 0$ up to the MCP; it is not clear whether the values of exponents are universal or not.

4.5. Absence of re-entrance

Here, we discuss the asymptotic behaviour of the auto-correlation function for process I with $t_{\mathrm{w}} = 0$, $\left[\langle \mathrm{e}^{\mathrm{i}\phi_i(t)-\mathrm{i}\phi_i(0)}\rangle_K^D\right]_{\mathrm{c}}$, which satisfies

$$\left[\langle \mathrm{e}^{\mathrm{i}\phi_i(t)}\rangle_D^{\mathbf{F}}\right]_{\mathrm{c}} = \left[\langle \mathrm{e}^{\mathrm{i}\phi_i(t)-\mathrm{i}\phi_i(0)}\rangle_K^D\right]_{\mathrm{c}} \tag{4.15}$$

to show the absence of re-entrant transition, i.e. the verticality of the FM boundary. We present the following two propositions for it. Since the long-time limit of this function is expected to behave as a kind of order parameter, (i) *the asymptotic behaviour should be prescribed by both*

phases where the initial point (D, D) *and the final point* (K, D) *are located*. In the appendix, we mention that the average of the gauge invariant function cannot detect the FM boundary. Because of the gauge invariance (3.27), (ii) *the asymptotic behaviour is independent of whether* (D, D) *and* (K, D) *are FM or not*. From this point of view, the behaviour of $\left[\langle \mathrm{e}^{\mathrm{i}\phi_i(t)-\mathrm{i}\phi_i(0)}\rangle_K^D\right]_\mathrm{c}$ can change only when (D, D) or (K, D) intersects the PM boundary. On the other hand, it approaches an FM order parameter because of equation (4.15), implying that the asymptotic behaviour changes depending on if (K, D) is FM or not. At a glance, this contradicts (ii) on the FM boundary below the PM boundary. However, they are not in conflict with each other if the line $K = D$ intersects the MCP and the FM phase boundary is vertical in the temperature below it. In this case, when (K, D) is located on the FM boundary, (D, D) is located on the PM boundary simultaneously (i.e. the MCP).

In two dimensions, the same argument can be made if one considers the asymptotic decay of relaxation functions as an order parameter for the KT phase instead of the long time limit of them. The order parameter relaxation $\left[\langle \mathrm{e}^{\mathrm{i}\phi_i(t)}\rangle_K^{\mathbf{F}}\right]_\mathrm{c}$ decays exponentially in the PM phase while it decays algebraically in the KT phase. Thus, the above argument is recognized as the statement that the KT boundary is vertical below the MCP.

5. Deterministic dynamics

In section 3, we treat the stochastic dynamics obeying the master equation with continuous time (3.1). The results can also be applied to the discrete time interval case, which is used in Monte Carlo simulations. Since the spin variable is continuous in the gauge glass model, one can define equations of motion. In this section, we discuss the case of the deterministic dynamics such as in molecular dynamics simulations.

It is noted that the essential features for a physical quantity Q to derive relations such as (3.32) and (3.33) are the gauge invariance

$$V_\psi \langle Q\rangle_K^D = \langle Q\rangle_K^D \tag{5.1}$$

for process I as equation (3.28), and the transformation rule

$$V_\psi \langle Q\rangle_K^{\mathbf{F}} = \langle Q\rangle_K^\psi \tag{5.2}$$

for process II as equation (3.29). By the use of equations (2.16) and (2.17), they reveal the relation

$$\left[\langle Q\rangle_K^{\mathbf{F}}\right]_\mathrm{c} = \left[\langle Q\rangle_K^D\right]_\mathrm{c} \tag{5.3}$$

irrespective of the detail of dynamics.

5.1. Microcanonical dynamics

Let us introduce a conjugate momentum L_i for each spin variable ϕ_i and consider the total Hamiltonian $\mathcal{H}_\mathrm{tot} = J\tilde{\mathcal{H}}_\mathrm{tot}(\mathbf{L}, \phi, \boldsymbol{\omega})$ including the kinetic energy

$$\tilde{\mathcal{H}}_\mathrm{tot}(\mathbf{L}, \phi, \boldsymbol{\omega}) = -\sum_{\langle ij\rangle} \cos(\phi_i - \phi_j + \omega_{ij}) + \frac{1}{2}\sum_i L_i^2. \tag{5.4}$$

The thermal distribution for this system is given by

$$\frac{\exp(-K\tilde{\mathcal{H}}_\mathrm{tot}(\mathbf{L}, \phi, \boldsymbol{\omega}))}{Z_\mathrm{tot}(K; \boldsymbol{\omega})} = \mu_\mathrm{eq}(\mathbf{L}; K)\rho_\mathrm{eq}(\phi; K, \boldsymbol{\omega}) \tag{5.5}$$

where the partition function of $\mathcal{H}_\mathrm{tot}$ is defined by

$$Z_\mathrm{tot}(K; \boldsymbol{\omega}) = \int \mathrm{d}\mathbf{L}\,\mathrm{d}\phi\, \exp(-K\tilde{\mathcal{H}}_\mathrm{tot}(\mathbf{L}, \phi, \boldsymbol{\omega})). \tag{5.6}$$

The integration of d$\mathbf{L}$ denotes multiple integrations for N momentum variables. For the spin variable, the thermal distribution of $\mathcal{H}_{\rm tot}$ is identical with that in the static case (2.2). The function $\mu_{\rm eq}(\mathbf{L}; K)$ is the thermal distribution for momentum variables

$$\mu_{\rm eq}(\mathbf{L}; K) \equiv \left(\frac{K}{2\pi}\right)^{N/2} \exp\left(-\frac{K}{2}\sum_i L_i^2\right). \tag{5.7}$$

The equations of motion for ϕ_i and L_i are given by the canonical equations of Hamilton

$$\frac{\partial L_i}{\partial t} = -\frac{\partial}{\partial \phi_i}\mathcal{H}_{\rm tot} = -J\sum_{(j\in i)} \sin(\phi_i - \phi_j + \omega_{ij}) \tag{5.8a}$$

$$\frac{\partial \phi_i}{\partial t} = \frac{\partial}{\partial L_i}\mathcal{H}_{\rm tot} = JL_i \tag{5.8b}$$

where the summation of $(j \in i)$ takes for all sites coupled with the i site.

Note that the conjugate momentum L_i, differential operators $\frac{\partial}{\partial L_i}$ and $\frac{\partial}{\partial \phi_i}$ are invariant under gauge transformations U_ψ and V_ψ. Thus, the canonical equations (5.8a) and (5.8b) are also invariant under the gauge transformation $U_\psi V_\psi$. This means that if $(\bar{\mathbf{L}}(t), \bar{\phi}(t))$ represents the solution for a bond configuration $\boldsymbol{\omega}$ with an initial condition $(\mathbf{L}_0, \phi_0)$, the solution for the bond configuration $V_\psi\boldsymbol{\omega}$ with the initial condition $(\mathbf{L}_0, \phi_0 - \psi)$ is given by $((\bar{\mathbf{L}}(t), \bar{\phi}(t) - \psi)$. This implies the transformation rules

$$U'_\psi V_\psi \bar{\phi}_i(t; \mathbf{L}_0, \phi_0, \boldsymbol{\omega}) = \bar{\phi}_i(t; \mathbf{L}_0, \phi_0, \boldsymbol{\omega}) - \psi_i \tag{5.9a}$$

$$U'_\psi V_\psi \bar{L}_i(t; \mathbf{L}_0, \phi_0, \boldsymbol{\omega}) = \bar{L}_i(t; \mathbf{L}_0, \phi_0, \boldsymbol{\omega}) \tag{5.9b}$$

where U'_ψ is the transformation operated to ϕ_0. It is remarked that once the transformation for ϕ_0 is operated, it affects $\phi(t)$ at any time, since the solution is continuous in time. Thus, the operation $U'_\psi V_\psi$ in equations (5.9a) and (5.9b) can be recognized as $U_\psi U'_\psi V_\psi$ in the stochastic dynamics.

Let us define dynamical averages such as equations (3.14)–(3.16). Since the dynamics is deterministic, the dynamical average is performed by the average over the initial condition $(\mathbf{L}_0, \phi_0)$. When the initial distribution is in equilibrium, the temperature is unchanged since the total energy is conserved. The equilibrium dynamics with temperature K, $\langle\cdots\rangle_K^{\rm eq} = \langle\cdots\rangle_K^K$, is defined as

$$\langle\cdots\rangle_K^{\rm eq} = \int {\rm d}\mathbf{L}_0\, {\rm d}\phi_0\, \mu_{\rm eq}(\mathbf{L}_0; K)\rho_{\rm eq}(\phi_0; K, \boldsymbol{\omega})\cdots. \tag{5.10}$$

Although the dynamics contains no dissipative factor in this formulation, we assume that the system shows a relaxation to an orbit which traces the thermal distribution when one prepares an appropriate nonequilibrium initial distribution $\rho_0(\mathbf{L}_0, \phi_0)$. This is related to the ergodic theorem. There are several ways to determine $\rho_0(\mathbf{L}_0, \phi_0)$. One of the simplest ways is to fix the total energy to the equilibrium value at the final temperature. In process I, $\langle\cdots\rangle_K^D$, the initial distribution for spin variables is the thermal distribution with temperature D. The total energy is identical to the equilibrium value at temperature K providing

$$\rho_{\rm I}(\mathbf{L}_0, \phi_0; K, D, \boldsymbol{\omega}) = \frac{\rho_{\rm eq}(\phi_0; D, \boldsymbol{\omega})}{\Omega_{\rm I}(K, D, \boldsymbol{\omega})}\delta(\tilde{\mathcal{H}}_{\rm tot}(\mathbf{L}_0, \phi_0, \boldsymbol{\omega}) - \langle\tilde{\mathcal{H}}_{\rm tot}\rangle_K) \tag{5.11}$$

where $\langle\tilde{\mathcal{H}}_{\rm tot}\rangle_K = N/2K + \langle\tilde{\mathcal{H}}\rangle_K$, and

$$\Omega_{\rm I}(K, D, \boldsymbol{\omega}) = \int {\rm d}\mathbf{L}_0\, {\rm d}\phi_0\, \rho_{\rm eq}(\phi_0; D, \boldsymbol{\omega})\delta(\tilde{\mathcal{H}}_{\rm tot}(\mathbf{L}_0, \phi_0, \boldsymbol{\omega}) - \langle\tilde{\mathcal{H}}_{\rm tot}\rangle_K) \tag{5.12}$$

is the volume of the phase space in process I. When $K = D$, the dynamical average with ρ_{I} becomes identical with (5.10), because of the equivalence of the canonical and microcanonical ensembles in large enough systems. Similarly, the initial distribution for process II′ is given by

$$\rho_\psi(\mathbf{L}_0, \phi_0; K, \boldsymbol{\omega}) = \frac{\delta(\phi_0 - \psi)}{\Omega_\psi(K, \boldsymbol{\omega})}\delta(\tilde{\mathcal{H}}_{\mathrm{tot}}(\mathbf{L}_0, \phi_0, \boldsymbol{\omega}) - \langle\tilde{\mathcal{H}}_{\mathrm{tot}}\rangle_K). \tag{5.13}$$

where

$$\Omega_\psi(K, \boldsymbol{\omega}) = \int \mathrm{d}\mathbf{L}_0\, \mathrm{d}\phi_0\, \delta(\phi_0 - \psi)\delta(\tilde{\mathcal{H}}_{\mathrm{tot}}(\mathbf{L}_0, \phi_0, \boldsymbol{\omega}) - \langle\tilde{\mathcal{H}}_{\mathrm{tot}}\rangle_K) \tag{5.14}$$

is the volume of the phase space in this process. Using these initial distributions, we define the correlation functions for processes I and II:

$$\langle \mathrm{e}^{\mathrm{i}\phi_i(t+t_{\mathrm{w}}) - \mathrm{i}\phi_i(t_{\mathrm{w}})}\rangle_K^D = \int \mathrm{d}\mathbf{L}_0\, \mathrm{d}\phi_0\, \rho_{\mathrm{I}}(\mathbf{L}_0, \phi_0; K, D, \boldsymbol{\omega})\, \mathrm{e}^{\mathrm{i}\bar{\phi}_i(t+t_{\mathrm{w}}; \mathbf{L}_0, \phi_0, \boldsymbol{\omega}) - \mathrm{i}\bar{\phi}_i(t_{\mathrm{w}}; \mathbf{L}_0, \phi_0, \boldsymbol{\omega})} \tag{5.15}$$

$$\langle \mathrm{e}^{\mathrm{i}\phi_i(t+t_{\mathrm{w}}) - \mathrm{i}\phi_i(t_{\mathrm{w}})}\rangle_K^{\mathbf{F}} = \int \mathrm{d}\mathbf{L}_0\, \mathrm{d}\phi_0\, \rho_{\mathbf{F}}(\mathbf{L}_0, \phi_0; K, \boldsymbol{\omega})\, \mathrm{e}^{\mathrm{i}\bar{\phi}_i(t+t_{\mathrm{w}}; \mathbf{L}_0, \phi_0, \boldsymbol{\omega}) - \mathrm{i}\bar{\phi}_i(t_{\mathrm{w}}; \mathbf{L}_0, \phi_0, \boldsymbol{\omega})}. \tag{5.16}$$

The dynamical averages for the energy are also defined in a similar way.

Noting the invariance (2.14)–(2.15), the function $\rho_{\mathrm{I}}(\mathbf{L}_0, \phi_0; K, D, \boldsymbol{\omega})$ is invariant under the gauge transformation $U'_\psi V_\psi$:

$$U'_\psi V_\psi \rho_{\mathrm{I}}(\mathbf{L}_0, \phi_0; K, D, \boldsymbol{\omega}) = \rho_{\mathrm{I}}(\mathbf{L}_0, \phi_0; K, D, \boldsymbol{\omega}). \tag{5.17}$$

This reveals the gauge invariance of $\langle \mathrm{e}^{\mathrm{i}\phi_i(t+t_{\mathrm{w}}) - \mathrm{i}\phi_i(t_{\mathrm{w}})}\rangle_K^D$ for process I as in the stochastic dynamics case (3.28). On the other hand, the transformation for $\rho_{\mathbf{F}}$ gives

$$U'_\psi V_\psi \rho_{\mathbf{F}}(\mathbf{L}_0, \phi_0; K, \boldsymbol{\omega}) = \rho_\psi(\mathbf{L}_0, \phi_0; K, \boldsymbol{\omega}) \tag{5.18}$$

which reveals the transformation rule for process II

$$\begin{aligned} V_\psi \langle \mathrm{e}^{\mathrm{i}\phi_i(t+t_{\mathrm{w}}) - \mathrm{i}\phi_i(t_{\mathrm{w}})}\rangle_K^{\mathbf{F}} &= \int \mathrm{d}\mathbf{L}_0\, \mathrm{d}\phi_0\, \rho_\psi(\mathbf{L}_0, \phi_0; K, \boldsymbol{\omega})\, \mathrm{e}^{\mathrm{i}\bar{\phi}_i(t+t_{\mathrm{w}}; \mathbf{L}_0, \phi_0, \boldsymbol{\omega}) - \mathrm{i}\bar{\phi}_i(t_{\mathrm{w}}; \mathbf{L}_0, \phi_0, \boldsymbol{\omega})} \\ &= \langle \mathrm{e}^{\mathrm{i}\phi_i(t+t_{\mathrm{w}}) - \mathrm{i}\phi_i(t_{\mathrm{w}})}\rangle_K^{\psi}. \end{aligned} \tag{5.19}$$

Thus, from equations (5.1)–(5.3), equation (5.19) yields equation (3.31) providing the aging relation (3.32). It is easy to see that equation (3.33) also holds.

5.2. Liouville equation

To compare the formulations in stochastic and deterministic dynamics discussed above, it is instructive to consider the classical Liouville equation which describes the time evolution of distribution function ρ_t under the canonical equations (5.8*a*) and (5.8*b*):

$$\frac{\mathrm{d}\rho_t}{\mathrm{d}t} = \sum_i \left\{ \frac{\partial \mathcal{H}_{\mathrm{tot}}}{\partial L_i}\frac{\partial \rho_t}{\partial \phi_i} - \frac{\partial \mathcal{H}_{\mathrm{tot}}}{\partial \phi_i}\frac{\partial \rho_t}{\partial L_i} \right\}. \tag{5.20}$$

If we define the solution of equation (5.20) for the initial condition $\rho = \delta(\mathbf{L} - \mathbf{L}_0)\delta(\phi - \phi_0)$ as $G(t, \mathbf{L}, \phi; \mathbf{L}_0, \phi_0, \boldsymbol{\omega})$, this is nothing but a delta function

$$G(t, \mathbf{L}, \phi; \mathbf{L}_0, \phi_0, \boldsymbol{\omega}) = \delta(\mathbf{L} - \bar{\mathbf{L}}(t; \mathbf{L}_0, \phi_0, \boldsymbol{\omega}))\delta(\phi - \bar{\phi}(t; \mathbf{L}_0, \phi_0, \boldsymbol{\omega})) \tag{5.21}$$

since the orbit is deterministic. The function G is called the Green function, and the solution for any initial distribution $\rho_0(\mathbf{L}, \phi)$ is given by

$$\rho_t(\mathbf{L}, \phi; \boldsymbol{\omega}, [\rho_0]) = \int \mathrm{d}\mathbf{L}_0\, \mathrm{d}\phi_0 G(t, \mathbf{L}, \phi; \mathbf{L}_0, \phi_0, \boldsymbol{\omega})\rho_0(\mathbf{L}_0, \phi_0) \tag{5.22}$$

where ρ_t is a functional of the initial distribution indicated in the arguments as $[\rho_0]$.

The Green function G plays a similar role with $\langle\phi|\,\mathrm{e}^{t\mathbf{W}}|\phi'\rangle$ in the stochastic dynamics. It is easy to see that equations (5.9*a*), (5.9*b*) and (5.21) yield the invariance

$$U_\psi U'_\psi V_\psi G(t, \mathbf{L}, \phi; \mathbf{L}_0, \phi_0, \boldsymbol{\omega}) = G(t, \mathbf{L}, \phi; \mathbf{L}_0, \phi_0, \boldsymbol{\omega}) \tag{5.23}$$

which is the same relation as equation (3.22). Since the differential operators and $\mathcal{H}_{\mathrm{tot}}$ is invariant under $U_\psi V_\psi$, when the function ρ_t is the solution of equation (5.20) with an initial condition ρ_0, the function $U_\psi V_\psi \rho_t$ is also the solution with the initial condition $U_\psi \rho_0$: it is easily confirmed from equation (5.22) that

$$U_\psi V_\psi \rho_t(\mathbf{L}, \phi; \boldsymbol{\omega}, [\rho_0]) = \rho_t(\mathbf{L}, \phi; \boldsymbol{\omega}, [U_\psi \rho_0]). \tag{5.24}$$

Using the definitions above with the Green function, the auto-correlation functions are expressed as

$$\begin{aligned}\langle \mathrm{e}^{\mathrm{i}\phi_i(t+t_\mathrm{w})-\mathrm{i}\phi_i(t_\mathrm{w})}\rangle_K^D = &\int \mathrm{d}\mathbf{L}\,\mathrm{d}\phi\,\mathrm{d}\mathbf{L}'\,\mathrm{d}\phi'\,\mathrm{d}\mathbf{L}_0\,\mathrm{d}\phi_0 G(t, \mathbf{L}, \phi; \mathbf{L}', \phi', \boldsymbol{\omega}) \\ &\times G(t_\mathrm{w}, \mathbf{L}', \phi'; \mathbf{L}_0, \phi_0, \boldsymbol{\omega})\rho_\mathrm{I}(\mathbf{L}_0, \phi_0; K, D, \boldsymbol{\omega})\,\mathrm{e}^{\mathrm{i}\phi_i - \mathrm{i}\phi'_i}\end{aligned} \tag{5.25}$$

for process I, and

$$\begin{aligned}\langle \mathrm{e}^{\mathrm{i}\phi_i(t+t_\mathrm{w})-\mathrm{i}\phi_i(t_\mathrm{w})}\rangle_K^{\mathbf{F}} = &\int \mathrm{d}\mathbf{L}\,\mathrm{d}\phi\,\mathrm{d}\mathbf{L}'\,\mathrm{d}\phi'\,\mathrm{d}\mathbf{L}_0\,\mathrm{d}\phi_0 G(t, \mathbf{L}, \phi; \mathbf{L}', \phi', \boldsymbol{\omega}) \\ &\times G(t_\mathrm{w}, \mathbf{L}', \phi'; \mathbf{L}_0, \phi_0, \boldsymbol{\omega})\rho_\mathbf{F}(\mathbf{L}_0, \phi_0; K, \boldsymbol{\omega})\,\mathrm{e}^{\mathrm{i}\phi_i - \mathrm{i}\phi'_i}.\end{aligned} \tag{5.26}$$

for process II. They are equivalent with equations (5.15) and (5.16). From invariance (5.23) and (5.17) with equation (2.11), one finds that $\langle \mathrm{e}^{\mathrm{i}\phi_i(t+t_\mathrm{w})-\mathrm{i}\phi_i(t_\mathrm{w})}\rangle_K^D$ is gauge invariant and the other one obeys

$$\begin{aligned}V_\psi\langle \mathrm{e}^{\mathrm{i}\phi_i(t+t_\mathrm{w})-\mathrm{i}\phi_i(t_\mathrm{w})}\rangle_K^{\mathbf{F}} = &\int \mathrm{d}\mathbf{L}\,\mathrm{d}\phi\,\mathrm{d}\mathbf{L}'\,\mathrm{d}\phi'\,\mathrm{d}\mathbf{L}_0\,\mathrm{d}\phi_0 G(t, \mathbf{L}, \phi; \mathbf{L}', \phi', \boldsymbol{\omega}) \\ &\times G(t_\mathrm{w}, \mathbf{L}', \phi'; \mathbf{L}_0, \phi_0, \boldsymbol{\omega})\rho_\psi(\mathbf{L}_0, \phi_0; K, \boldsymbol{\omega})\,\mathrm{e}^{\mathrm{i}\phi_i - \mathrm{i}\phi'_i}.\end{aligned} \tag{5.27}$$

Thus, from equations (5.1)–(5.3), this yields equation (3.31) providing the aging relation (3.32).

5.3. Canonical initial distribution

Another way of defining the initial distributions instead of (5.11) and (5.13) is given by the canonical ensemble. In both processes, the initial distribution is determined so that the total energy distributes in terms of the thermal distribution with temperature K:

$$\int \mathrm{d}\mathbf{L}_0\,\mathrm{d}\phi_0\,\rho_0(\mathbf{L}_0, \phi_0)\delta(E - \mathcal{H}_{\mathrm{tot}}(\mathbf{L}_0, \phi_0, \boldsymbol{\omega})) = \frac{\mathcal{D}_{\mathrm{tot}}(E; \boldsymbol{\omega})\,\mathrm{e}^{-KE/J}}{Z_{\mathrm{tot}}(K; \boldsymbol{\omega})}. \tag{5.28}$$

where $\mathcal{D}_{\mathrm{tot}}(E; \boldsymbol{\omega})$ is the density of state

$$\mathcal{D}_{\mathrm{tot}}(E; \boldsymbol{\omega}) = \int \mathrm{d}\mathbf{L}_0\,\mathrm{d}\phi_0\,\delta(E - \mathcal{H}_{\mathrm{tot}}). \tag{5.29}$$

For process I, ρ_0 has the form

$$\rho_\mathrm{I}(\mathbf{L}_0, \phi_0) = \mu_\mathrm{I}(\mathbf{L}_0, \phi_0; K, D, \boldsymbol{\omega})\rho_{\mathrm{eq}}(\phi_0; D, \boldsymbol{\omega}). \tag{5.30}$$

Equation (5.28) implicitly defines the function $\mu_\mathrm{I}(\mathbf{L}_0, \phi_0; K, D, \boldsymbol{\omega})$ depending on K, D and $\boldsymbol{\omega}$. For the case $K = D$, $\mu_\mathrm{I} = \mu_{\mathrm{eq}}(\mathbf{L}_0; D)$ satisfies equation (5.28). For process II′, ρ_0 has the form

$$\rho_\psi(\mathbf{L}_0, \phi_0) = \mu_\psi(\mathbf{L}_0, \phi_0; K, \boldsymbol{\omega})\delta(\phi_0 - \psi). \tag{5.31}$$

Equation (5.28) implicitly defines the function $\mu_\psi(\mathbf{L}_0, \phi_0; K, \boldsymbol{\omega})$ depending on K and $\boldsymbol{\omega}$.

Noting the invariance (2.9), (2.13) and (2.14) with the invariant integral (2.11), one finds that, if $\rho_0(\mathbf{L}_0, \phi_0)$ satisfies equation (5.28), so does $U'_\psi V_\psi \rho_0(\mathbf{L}_0, \phi_0)$. Since $U'_\psi V_\psi \mu_{\mathrm{I}}$ instead of μ_{I} satisfies the same equation as (5.28) with (5.30), the function $\rho_{\mathrm{I}}(\mathbf{L}_0, \phi_0)$ is invariant under $U'_\psi V_\psi$. This reveals the gauge invariance of $\langle \mathrm{e}^{\mathrm{i}\phi_i(t+t_{\mathrm{w}})-\mathrm{i}\phi_i(t_{\mathrm{w}})}\rangle_K^D$ for process I. On the other hand, if $\rho_{\mathbf{F}}(\mathbf{L}_0, \phi_0)$ satisfies equation (5.28) for process II, so does

$$U'_\psi V_\psi \rho_{\mathbf{F}}(\mathbf{L}_0, \phi_0) = [U'_\psi V_\psi \mu_{\mathbf{F}}(\mathbf{L}_0, \phi_0; K, \boldsymbol{\omega})]\delta(\phi_0 - \psi). \tag{5.32}$$

This means

$$U'_\psi V_\psi \rho_{\mathbf{F}}(\mathbf{L}_0, \phi_0; K, \boldsymbol{\omega}) = \rho_\psi(\mathbf{L}_0, \phi_0; K, \boldsymbol{\omega}). \tag{5.33}$$

Therefore one can derive the transformation rules such as (5.19) providing (3.31) and the aging relation (3.32) for the canonical case.

5.4. Extended system

In the above, we have defined relaxation processes in conservative dynamics with gauge symmetry. Here we consider another deterministic dynamics, which keeps the temperature constant [72, 73]. It is called the extended system dynamics.

In addition to the momentum variables, we introduce a single degree of freedom, the daemon, which couples to the original system. It will act as a heat bath for the original system and convert the dynamics as a fixed temperature one. Let the variable y be the daemon and ℓ the conjugate momentum of it. The Hamiltonian of the extended system $\mathcal{H}_{\mathrm{ext}} = J\tilde{\mathcal{H}}_{\mathrm{ext}}(\tilde{\mathbf{L}}, \tilde{\phi}, \boldsymbol{\omega})$ is defined by

$$\tilde{\mathcal{H}}_{\mathrm{ext}}(\tilde{\mathbf{L}}, \tilde{\phi}, \boldsymbol{\omega}) = -\sum_{\langle ij\rangle} \cos(\phi_i - \phi_j + \omega_{ij}) + \frac{1}{2y^2}\sum_i L_i^2 + \frac{\ell^2}{2} + \frac{(N+1)}{K}\ln y \tag{5.34}$$

where $\tilde{\phi}$ represents the set of spin variables ϕ including the daemon y, and $\tilde{\mathbf{L}}$ represents the set of momentum variables $\mathbf{L}$ including ℓ. Note that the temperature K is explicitly included in $\mathcal{H}_{\mathrm{ext}}$. The canonical equations are given by

$$\frac{\partial L_i}{\partial t} = -J\sum_{(j\in i)} \sin(\phi_i - \phi_j + \omega_{ij}) \tag{5.35a}$$

$$\frac{\partial \phi_i}{\partial t} = JL_i/y^2 \tag{5.35b}$$

$$\frac{\partial \ell}{\partial t} = J\left(\sum_i L_i^2/y^2 - (N+1)/K\right)\Big/ y \tag{5.35c}$$

$$\frac{\partial y}{\partial t} = J\ell. \tag{5.35d}$$

It is known that the microcanonical ensemble of $\mathcal{H}_{\mathrm{ext}}$ with the total energy E becomes the canonical ensemble of $\mathcal{H}_{\mathrm{tot}}$ with temperature K irrespective of E when one traces out the daemon variables y and ℓ. This means that the solution of the equations with an arbitrary initial condition shows a relaxation to an orbit which traces the equilibrium distribution with temperature K, if one assumes the ergodic theorem.

In this system, the dynamical averages are given by

$$\langle\cdots\rangle_K^D = \int \mathrm{d}\tilde{\mathbf{L}}_0\,\mathrm{d}\tilde{\phi}_0\,\mu(\mathbf{L}_0, \ell, y)\rho_{\mathrm{eq}}(\phi_0; D, \boldsymbol{\omega})\cdots \tag{5.36}$$

for process I and

$$\langle\cdots\rangle_K^{\mathbf{F}} = \int \mathrm{d}\tilde{\mathbf{L}}_0\,\mathrm{d}\tilde{\phi}_0\,\mu(\mathbf{L}_0,\ell,y)\delta(\phi_0-\mathbf{F})\cdots \tag{5.37}$$

for process II, where the integrations of $\mathrm{d}\tilde{\mathbf{L}}_0$ and $\mathrm{d}\tilde{\phi}_0$ denote the multiple integrations for $N+1$ momentum variables including the daemon's ℓ and those for $N+1$ spin variables including y, respectively. As stated above, the initial distribution function $\mu(\mathbf{L}_0,\ell,y)$ for $\mathbf{L}$, ℓ and y can be chosen arbitrarily. We assume that $\mu(\mathbf{L}_0,\ell,y)$ is independent of $\boldsymbol{\omega}$ so that it is gauge invariant. It is natural to consider that the daemon variables y and ℓ are invariant under the gauge transformation $U_\psi V_\psi$. Thus, the canonical equations (5.35) are also invariant under the transformation, which reveals a transformation rule like (5.9):

$$U'_\psi V_\psi \bar{\phi}_i(t;\tilde{\mathbf{L}}_0,\tilde{\phi}_0,\boldsymbol{\omega}) = \bar{\phi}_i(t;\tilde{\mathbf{L}}_0,\tilde{\phi}_0,\boldsymbol{\omega}) - \psi_i. \tag{5.38}$$

Following the previous subsection, one can show the gauge invariance of $\langle \mathrm{e}^{\mathrm{i}\phi_i(t+t_\mathrm{w})-\mathrm{i}\phi_i(t_\mathrm{w})}\rangle_K^D$ for process I and the transformation rule such as (5.19) for process II providing (3.31) and the aging relation (3.32) for the extended system.

6. Summary

We have applied the gauge transformation to dynamical systems of the gauge glass. The stochastic dynamics with the master equation formalism (3.1) as well as the deterministic one with the canonical equations of Hamilton are treated. For the deterministic case, we consider standard equations of motion with no dissipative factor ((5.8*a*) and (5.8*b*)), in which one realizes relaxation processes to choose appropriate initial distributions. Both microcanonical and canonical ensembles are considered for the initial distribution. Further, we consider the extended system dynamics ((5.35*a*)–(5.35*d*)) in which the temperature is fixed. In this dynamics, a single degree of freedom is introduced, which is called the daemon and plays the role of a heat bath. For all dynamics, we show the theory of gauge transformation in a coherent manner.

The dynamical relations (3.32) and (3.33) are derived exactly for all dynamics. The equivalence of nonequilibrium relaxations in the SG phase has been shown between the ZFC and FQ processes. The waiting-time dependence, which is typical for the aging phenomenon, coincides for ZFC and FQ processes. On the line $K = D$, we have found that the equilibrium relaxation coincides with the nonequilibrium one from the strong field limit. This indicates the absence of aging providing the restrictions for the regime of possible mixed phase and Griffiths phase. The exact relations for critical exponents are found at the MCP on this line. Further, we have confirmed the verticality of the FM phase boundary or equivalently the absence of the re-entrant transition.

In contrast to the standard SG systems, not many studies on the dynamical properties have been performed for gauge glass systems. The present results would be helpful for future investigations including dynamics.

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Appendix. Modified model

The modified model [20, 44] is useful to consider the qualitative behaviours of physical quantities and the topology of the phase diagram. The Hamiltonian is identical with the original model. The distribution of randomness, including a fixed constant a, is modified from the original model ($a = 0$ case),

$$P_a(\boldsymbol{\omega}; D) \equiv P(\boldsymbol{\omega}; D + a)\frac{Y(D + a)Z(D; \boldsymbol{\omega})}{Y(D)Z(D + a; \boldsymbol{\omega})}. \tag{A1}$$

The random average in the modified model with a is denoted by

$$\{\cdots\}_{\mathrm{c}}^{a} \equiv \int \mathrm{d}\boldsymbol{\omega}\, P_a(\boldsymbol{\omega}; D) \cdots . \tag{A2}$$

It is shown [20] that the average of any gauge invariant quantity is independent of a, i.e.

$$\{Q(\boldsymbol{\omega})\}_{\mathrm{c}}^{a} = [Q(\boldsymbol{\omega})]_{\mathrm{c}}. \tag{A3}$$

In this appendix, we summarize the properties of the modified model obtained previously. Analysing the FM and SG order parameters defined from the static FM and SG correlation functions,

$$m_a(K, D)^2 \equiv \lim_{R_{ij}\to\infty} \left|\{\langle \mathrm{e}^{\mathrm{i}\phi_i - \mathrm{i}\phi_j}\rangle_K\}_{\mathrm{c}}^{a}\right| \tag{A4}$$

$$q_a(K, D)^2 \equiv \lim_{R_{ij}\to\infty} \left\{|\langle \mathrm{e}^{\mathrm{i}\phi_i - \mathrm{i}\phi_j}\rangle_K|^2\right\}_{\mathrm{c}}^{a} \tag{A5}$$

the following properties have been found exactly for the model with a:

- The PM boundary, at which the edge of $q_a = 0$ is located, is unchanged with a.
- The ordered phase on $K = D + a$ must be the FM phase, since $m_a = q_a$ holds on it.
- The line $K = D + a$ is likely to intersects the MCP.
- The FM boundary below $K = D + a$ is vertical or re-entrant. Further, verticality can be shown if the ordered phase between the line $K = D + a$ and the non-random case ($D = \infty$) is always FM for the modified model with any a; this assumption is quite plausible since both sides exhibit only PM–FM transition.

These properties are identical to those for the original model ($a = 0$). In two dimensions, one can treat KT transitions in the same way by analysing the correlation length for FM and SG orderings

$$\xi_a(K, D) \equiv \lim_{R_{ij}\to\infty} \left|\frac{\partial}{\partial R_{ij}} \ln \left|\left\{\langle \mathrm{e}^{\mathrm{i}\phi_i - \mathrm{i}\phi_j}\rangle_K\right\}_{\mathrm{c}}^{a}\right|\right|^{-1} \tag{A6}$$

$$\tilde{\xi}_a(K, D) \equiv \lim_{R_{ij}\to\infty} \left|\frac{\partial}{\partial R_{ij}} \ln \left\{\left|\langle \mathrm{e}^{\mathrm{i}\phi_i - \mathrm{i}\phi_j}\rangle_K\right|^2\right\}_{\mathrm{c}}^{a}\right|^{-1} \tag{A7}$$

instead of order parameters (A4) and (A5).

Since the FM correlation function as well as the FM and KT order parameters equations (A4) and (A6) are not gauge invariant, the FM (KT) boundary changes with a: at least on $K = D + a$, the FM (KT) regime changes with a. Thus, the qualitative behaviour of averaged gauge invariant quantities at (K, D) should not be influenced by the fact whether (K, D) is FM (KT) or not. In other words, any gauge invariant quantity cannot be an order parameter for the FM (KT) phase in modified models including the original one ($a = 0$).

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